The time evolution of a flat universe with a non-vanishing cosmological constant

Chapter 1

Late times: a flat ΛCDM universe, neglecting radiation

1.1 Governing differential equation

For $k = 0$ (flat universe), the first Friedmann-Lemaitre equation can be written as:

$$H^2 = \frac{8\pi G}{3}(\rho + \lambda), \tag{1.1}$$

where H is the Hubble parameter, G is the gravitational constant, ρ the density of matter and radiation (including Dark Matter), and λ the cosmological constant. Taking the time derivative and paying attention to λ being constant yields:

$$\dot{H} = \frac{4\pi G}{3}\frac{\dot{\rho}}{H} \tag{1.2}$$

The density and pressure of the radiation is negligible (except in the early stages of the universe!); therefore $p = 0$, and the second Friedmann-Lemaitre equation gives:

$$\dot{\rho} = -3H\rho.$$

Plugging this into (1.2) yields:

$$\dot{H} = -4\pi G\rho$$

Using for ρ now again the first Friedmann-Lemaitre equation, one obtains the differential equation

$$-\frac{2}{3}\dot{H} = H^2 - \frac{8\pi G}{3}\lambda$$

We'll see below that for $t \to \infty$, we have $H = H_{lim} =$ const. and hence $\dot{H} = 0$. This yields:

$$H_{lim}^2 = \frac{8\pi G}{3}\lambda,$$

and hence the differential equation become simply

$$-\frac{2}{3}\dot{H} = H^2 - H_{lim}^2.$$

We now define the critical density

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

and the density parameter for the cosmological constant:

$$\Omega_\lambda = \frac{\lambda}{\rho_{crit}} = \frac{8\pi G}{3H^2}\lambda.$$

That enables us to write:

$$H_{lim}^2 = \Omega_\lambda H^2 = \Omega_{\lambda,0} H_0^2,$$

where the subscript "0" indicates that these are the values at the time t_0, i.e. our current time. This relation holds because $H_{lim} = const.$

1.2 Time dependence of the Hubble parameter

One can rewrite the differential equation for the Hubble parameter as

$$\frac{dH/H_{lim}}{1 - (H/H_{lim})^2} = \frac{dt}{t_{ch}}$$

with the characteristic time scale

$$t_{ch} = \frac{2}{3H_{lim}}.$$

Integrating then leads to:

$$\left[\operatorname{arcoth}\left(\frac{H}{H_{lim}}\right)\right]_{t_0}^{t} = \frac{t - t_0}{t_{ch}}$$

(The integration gives also a second possible solution, with the area tangens hyperbolicus instead of the area cotangens hyperbolicus; essentially one has to use the boundary conditions to choose the right solution.)

At the Big Bang ($t = 0$), by definition $H \to \infty$ had to hold. This gives the first important result, the time t_0 since the Big Bang (using the connection between H_{lim} and $\Omega_{\lambda,0}$):

$$\boxed{t_0 = t_{ch} \operatorname{artanh}\left(\frac{H_{lim}}{H_0}\right) = t_{ch} \operatorname{artanh}\left(\sqrt{\Omega_{\lambda,0}}\right)} \qquad (1.3)$$

(note that taking the radiation into account would not change this result significantly, since the radiation dominated epoch lasted only a comparatively short time), and the formula for the Hubble parameter becomes then simply:

$$\boxed{H(t) = H_{lim} \coth\left(\frac{t}{t_{ch}}\right).}$$

Here we see what was already said at the end of the last section: H_{lim} is the limit of H for $t \to \infty$.

1.3 Time dependence of the scale parameter

For the scale parameter (resp. the curvature radius), we have the following differential equation:

$$H = \frac{\dot{a}}{a}.$$

Using the result for $H(t)$ obtained above, this implies

$$a(t) \sim \left(\sinh\left(\frac{t}{t_{ch}}\right)\right)^{2/3}.$$

Using the normalization $a(t_0) = 1$, we get:

$$a(t) = \left(\frac{\sinh\left(\frac{t}{t_{ch}}\right)}{\sinh\left(\frac{t_0}{t_{ch}}\right)}\right)^{2/3},$$

and by inserting the formula (1.3) for t_0, this simplifies to:

$$\boxed{a(t) = \left(\frac{1-\Omega_{\lambda,0}}{\Omega_{\lambda,0}}\right)^{1/3}\left(\sinh\left(\frac{t}{t_{ch}}\right)\right)^{2/3}.} \tag{1.4}$$

Taking into account that $\frac{1-\Omega_\lambda}{\Omega_\lambda} = \frac{\rho}{\lambda} \sim a^{-3}$, one sees that

$$\left(\frac{1-\Omega_{\lambda,0}}{\Omega_{\lambda,0}}\right)^{1/3} = a_{eq},$$

where a_{eq} is the scale factor at the time when the matter and dark energy densities were equal, and hence

$$a(t) = a_{eq}\left(\sinh\left(\frac{t}{t_{ch}}\right)\right)^{2/3}.$$

This also yields the result

$$t_{eq} = t_{ch}\mathrm{arsinh}(1).$$

Taking the second time derivative of the scale factor and setting it equal to zero, we obtain that the transition from decelerated to accelerated expansion (the "cosmic jerk") happened at the time

$$t_{jerk} = t_{ch}\mathrm{artanh}\left(\sqrt{\frac{1}{3}}\right),$$

hence:

$$\frac{t_{jerk}}{t_0} = \frac{\mathrm{artanh}\left(\sqrt{\frac{1}{3}}\right)}{\mathrm{artanh}\left(\sqrt{\Omega_{\lambda,0}}\right)}$$

and

$$\frac{t_{eq}}{t_{jerk}} = \frac{\mathrm{arsinh}(1)}{\mathrm{artanh}\left(\sqrt{\frac{1}{3}}\right)} \approx 1.34$$

1.4 Red shift

The red shift z obeys the following formula:

$$1 + z(t) = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}.$$

Plugging in (1.4) yields:

$$1 + z(t) = \frac{1}{a_{eq}\left(\sinh\left(\frac{t}{t_{ch}}\right)\right)^{2/3}}.$$

Inserting the result for the time of the "cosmic jerk" from above, one arrives at this result for the redshift at which the jerk happened:

$$1 + z_{jerk} = \frac{2^{1/3}}{a_{eq}}$$

This also implies

$$\frac{1 + z_{jerk}}{1 + z_{eq}} = 2^{1/3} \approx 1.26$$

A further important result is the relationship between H and z:

$$H(z) = H_0\sqrt{(1 - \Omega_{\lambda,0})(1 + z)^3 + \Omega_{\lambda,0}}$$

Finally, we also arrive at the following general relationship between the red shift of an object and the time when the light we see from it was emitted:

$$\boxed{t = t_{ch}\mathrm{arsinh}\left(\frac{1}{\sqrt{(a_{eq}\cdot(1 + z))^3}}\right).}$$

Note that this formula is only valid for not too large redshifts; e.g. the time of the decoupling ($z \approx 1089$) comes out wrong here, since at that time, the radiation was not negligible.

Light travels on null geodesics; therefore the distance traveled by light between the times t_1 and t_2 is

$$D = c\int_{t_1}^{t_2} \frac{dt}{a(t)}.$$

Inserting the $a(t)$ from (1.4) and using the substitution $x = t/t_{ch}$ leads then to

$$D = \frac{2c}{3H_0}\left(\sqrt{\Omega_{\lambda,0}}(1 - \Omega_{\lambda,0})\right)^{-1/3}\int_{x_1}^{x_2} \sinh(x)^{-2/3}dx.$$

If one wants to know which distance light traveled which was emitted at redshift z to us, one has to use

$$\begin{aligned} x_1 &= \mathrm{arsinh}\left(\frac{1}{\sqrt{(a_{eq}\cdot(1 + z))^3}}\right) \\ x_2 &= \mathrm{artanh}\left(\sqrt{\Omega_{\lambda,0}}\right) \end{aligned}$$

The remaining integral can then be done e.g. using Mathematica.

1.5 The density parameters

Finally, for the time evolution of the density parameters for the cosmological constant and for the matter we obtain:

$$\boxed{\Omega_\lambda = \Omega_{\lambda,0} \frac{H_0^2}{H^2} = \tanh^2\left(\frac{t}{t_{ch}}\right).}$$

$$\boxed{\Omega_\rho = 1 - \Omega_\lambda = \cosh^{-2}\left(\frac{t}{t_{ch}}\right).}$$

Especially, this yields

$$\Omega_{\lambda,jerk} = \frac{1}{3}.$$

If one wants to have an equation of state for the whole universe, one has to write

$$p(t) = w(t)\rho(t).$$

The total density is the density of matter plus the density of the dark energy; the total pressure is (to a good approximation) identical with the pressure of the dark energy, which is equal to minus its density. Hence we have:

$$-\rho_\lambda(t) = w(t)(\rho_M(t) + \rho_\lambda(t))$$

or

$$w(t) = -\frac{\Omega_\lambda(t)}{\Omega_{matter}(t) + \Omega_\lambda(t)}$$

But the denominator gives simply 1, hence:

$$w(t) = -\tanh^2\left(\frac{t}{t_{ch}}\right).$$

Chapter 2

Early times: a flat CDM universe with radiation, neglecting Λ

2.1 Governing differential equation

Now we have for the total density the sum of the matter and the radiation density:

$$\rho = \rho_M + \rho_R,$$

and for the radiation,

$$p_r = \frac{1}{3}\rho_R c^2$$

holds. Using the second Friedmann-Lemaitre equation then gives:

$$\begin{aligned}\dot{\rho}_M &= -3H\rho_M \\ \dot{\rho}_R &= -4H\rho_R.\end{aligned}$$

Using

$$H = \frac{\dot{a}}{a} \tag{2.1}$$

and the normalisation

$$a(t_0) = 1,$$

the solutions for the differential equations for the densities are

$$\begin{aligned}\rho_M &= \rho_{M,0}a^{-3} \\ \rho_R &= \rho_{R,0}a^{-4},\end{aligned}$$

where $\rho_{M,0}$ and $\rho_{R,0}$ are the values of the densities at time t_0. Since $\rho = \rho_{crit}\Omega = \frac{3H^2}{8\pi G}\Omega$, it follows that

$$\begin{aligned}H^2\Omega_M &= H_0^2\Omega_{M,0}a^{-3} \\ H^2\Omega_R &= H_0^2\Omega_{R,0}a^{-4}\end{aligned}$$

Now we can write (using flatness and neglecting the contribution from the cosmological constant):

$$H^2 = H^2(\Omega_M + \Omega_R) = H_0^2(\Omega_{M,0}a^{-3} + \Omega_{R,0}a^{-4}),$$

and using (2.1), this yields the following differential equation for the scale parameter:

$$a\dot{a} = H_0\sqrt{\Omega_{M,0}a + \Omega_{R,0}} = H_0\sqrt{\Omega_{M,0}}\sqrt{a + \frac{\Omega_{R,0}}{\Omega_{M,0}}} \tag{2.2}$$

Then introduce the abbreviation

$$\eta = \frac{\rho_R}{\rho_M} = \frac{\Omega_R}{\Omega_M}$$

(usually, η is used for the ratio of the *numbers* of photons and baryons - don't confuse that with the definition used here!). Inserting this relation in (2.2) above, we now get the differential equation

$$a\dot{a} = H_0\sqrt{\Omega_{M,0}}\sqrt{a + \eta_0}$$

2.2 Time dependence of the scale parameter

Solving the differential equation above (with the additional condition that $a = 0$ for $t = 0$) yields (see Bronstein, integral 125)

$$\frac{2}{3}(a - 2\eta_0)\sqrt{a + \eta_0} + \frac{4}{3}\eta_0\sqrt{\eta_0} = \sqrt{\Omega_{M,0}}H_0 t.$$

Solving this equation for the time yields:

$$t = \frac{2}{3H_0\sqrt{\Omega_{M,0}}}\left[(a - 2\eta_0)\sqrt{a + \eta_0} + 2\eta_0^{3/2}\right]. \tag{2.3}$$

Defining the characteristic time scale

$$t_{ch} = \frac{2\eta_0^{3/2}}{3H_0\sqrt{\Omega_{M,0}}}$$

enables us to express this in terms of the ratio of the scale parameter and the density ratio parameter:

$$\frac{t}{t_{ch}} = \left(\frac{a}{\eta_0} - 2\right)\sqrt{\frac{a}{\eta_0} + 1} + 2.$$

Solving the equation above for the scale parameter is a lot more complicated... First, we write

$$b = \frac{a}{\eta_0} - 1$$

and

$$\tau = \frac{t}{t_{ch}} - 2.$$

This yields the much simpler equation

$$(b-1)\sqrt{b+2} = \tau. \tag{2.4}$$

Note that due to $a \geq 0$ and $t \geq 0$, the new variables have to satisfy the constraints $b \geq -1$ and $\tau \geq -2$.

Squaring the equation (2.4), we arrive at the reduced cubic equation

$$b^3 - 3b + 2 - \tau^2 = 0 \tag{2.5}$$

with the determinant

$$D = \frac{1}{4}(\tau+2)\tau^2(\tau-2).$$

We now first study the three special cases with $D = 0$ (for which (2.5) has one single and one double real solution):

1. $\tau = -2$ yields the solutions $b = -1$ and $b = 2$ for (2.5), but only the first one also solves 2.4

2. $\tau = 0$ yields the solutions $b = -2$ and $b = 1$ for (2.5), but due to the constraint mentioned above, only the second one can be used

3. $\tau = 2$ yields the solutions $b = -1$ and $b = 2$ for (2.5), but only the second one also solves 2.4

In order to solve the equation in the intervals where $D \neq 0$, we have to use the formulas by Cardano. First, for $\tau > 2$, we have $D > 0$ and hence only one single real solution for (2.5). Then

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} = \sqrt[3]{\frac{1}{2}\tau^2 - 1 + \frac{1}{2}\tau\sqrt{\tau^2-4}}.$$

is real, and we have

$$v = -\frac{p}{3u} = \frac{1}{u}$$

also real. Hence the real solution is the first one of Cardano's three solutions:

$$\boxed{b(\tau) = u + \frac{1}{u} = \sqrt[3]{\frac{1}{2}\tau^2 - 1 + \frac{1}{2}\tau\sqrt{\tau^2-4}} + \frac{1}{\sqrt[3]{\frac{1}{2}\tau^2 - 1 + \frac{1}{2}\tau\sqrt{\tau^2-4}}}.}$$

for $\tau > 2$.

For $-2 < \tau < 0$ and $0 < \tau < 2$, we have $D < 0$ and hence three real solutions for (2.5).

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} = \sqrt[3]{\frac{1}{2}\tau^2 - 1 + \frac{i}{2}|\tau|\sqrt{4-\tau^2}}$$

is now complex, and for v we have

$$v = -\frac{p}{3u} = \frac{1}{u} = \frac{u^*}{|u|^2} = u^*,$$

because

$$
\begin{aligned}
|u|^2 &= \left|\sqrt[3]{\frac{1}{2}\tau^2 - 1 + \frac{i}{2}|\tau|\sqrt{4-\tau^2}}\right|^2 = \sqrt[3]{\left|\frac{1}{2}\tau^2 - 1 + \frac{i}{2}|\tau|\sqrt{4-\tau^2}\right|^2} \\
&= \sqrt[3]{\left(\frac{1}{2}\tau^2 - 1\right)^2 + \left(\frac{1}{2}|\tau|\sqrt{4-\tau^2}\right)^2} = \sqrt[3]{\frac{1}{4}\tau^4 - \tau^2 + 1 + \frac{1}{4}\tau^2(4-\tau^2)} \\
&= 1.
\end{aligned}
$$

Hence Cardano's three solutions are:

$$
\begin{aligned}
b_1 &= 2 \text{ Re } u \\
b_{2,3} &= -\text{Re } u \pm \sqrt{3} \text{ Im } u.
\end{aligned}
$$

In order to find out which solution we have to use, we now examine u more closely. The argument of the cubic root is a complex number with magnitude 1 (see above) whose imaginary part is always positive. Hence the argument of the cubic root lies on the upper half unit circle. This implies that u is a complex number with magnitude 1 and a phase between 0 and $\pi/3$. It follows that:

$$
\begin{aligned}
1 \leq & \; b_1 \leq 2 \\
-1 \leq & \; b_2 \leq 1 \\
-2 \leq & \; b_3 \leq -1.
\end{aligned}
$$

Comparing this with the solutions for $\tau = \pm 2$ and $\tau = 0$ above, we see that we have to use

- $b(\tau) = b_2 = -\text{Re } u + \sqrt{3} \text{ Im } u$ for $-2 < \tau < 0$, and
- $b(\tau) = b_1 = 2 \text{ Re } u$ for $0 < \tau < 2$.

It remains to express the real and imaginary parts with elementary functions. For that, we note that one can express the real and imaginary parts of any complex number z of magnitude 1 by

$$
\begin{aligned}
\text{Re } z &= \cos\varphi \\
\text{Im } z &= \sin\varphi,
\end{aligned}
$$

where φ is the phase of z. It then follows that

$$
\begin{aligned}
\text{Re } \sqrt[3]{z} &= \cos\left(\frac{\varphi}{3}\right) \\
\text{Im } \sqrt[3]{z} &= \sin\left(\frac{\varphi}{3}\right).
\end{aligned}
$$

Employing these relations, the solution for $\tau \leq 2$ is then given by:

$$
\begin{aligned}
b(\tau) &= -\cos\left(\frac{1}{3}\arccos(\tau^2 - 2)\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos(\tau^2-2)\right) \quad \text{for } -2 \leq \tau \leq 0 \\
b(\tau) &= 2\cos\left(\frac{1}{3}\arccos(\tau^2-2)\right) \quad \text{for } 0 \leq \tau \leq 2.
\end{aligned}
$$

Using

$$
a(t) = \eta_0 \left[b\left(t_{ch}(\tau+2)\right) + 1\right]
$$

yields then finally the respective solutions for $a(t)$.

2.3 Value of the density ratio parameter today

In order to get meaningful results, we now have the know the value of η_0.

For the density of radiation, one only has to consider the part of the radiation which was already present in the early universe (until about the time of the decoupling, which was considerably later than the radiation-dominated epoch), since all radiation produced later had essentially no influence on the evolution of the universe. But that part is essentially the CMBR and thermal neutrinos (other forms of radiation were only important at an even earlier epoch of the universe), i. e. blackbody radiation. The energy density of blackbody radiation is given by

$$\rho_{blackbody} = \frac{g\pi^2}{30\hbar^3 c^5}(kT)^4,$$

where g is the number of degrees of freedom. Additionally, one has to take into account that while T is here the temperature of the CMBR, T_{CMBR}, the neutrinos have a lower temperature (?)

$$\frac{T_\nu}{T_{CMBR}} = \left(\frac{4}{11}\right)^{1/3}$$

and give only an "effective massless number" of degrees of freedom" (?): every neutrino degree of freedom contributes only with a factor of 7/8 to the total number of degrees of freedom. The number of degrees of freedom for all three neutrinos and anti-neutrinos is then given by $7/8 \cdot 3 \cdot 2 = 21/4$. Hence the total radiation density is given by

$$\rho_{R,0} = \frac{\pi^2(2 + 21/4 \cdot (4/11)^{4/3})}{30\hbar^3 c^5}(kT_{CMBR})^4$$

(see http://pdg.lbl.gov/1998/bigbangrpp.pdf), and hence we get,

$$\Omega_{R,0} = \frac{\rho_{R,0}}{\rho_{crit,0}} = \frac{8\pi^3 G(2 + 21/4 \cdot (4/11)^{4/3})(kT_{CMBR})^4}{90\hbar^3 c^5 H_0^2} \approx \frac{7.537 \cdot 10^{-7}}{h_0^2}\frac{T_{CMBR}^4}{K^4}$$

with $H_0 = h_0 \cdot$ 100 km/s/Mpc $\approx h_0 \cdot 3.24 \cdot 10^{-18}$ 1/s or

$$\eta_0 \approx \frac{7.537 \cdot 10^{-7}}{h_0^2 \Omega_{M,0}}\frac{T^4}{K^4}$$

2.4 Results

Inserting the accepted value $T_{CMBR} = 2.72528$ K, this gives

$$\eta_0 \approx \frac{4.1576 \cdot 10^{-5}}{h_0^2 \Omega_{M,0}},$$

Using standard values from the WMAP satellite ($h_0^2\Omega_{M,0} = 0.135$) yields then

$$\eta_0 \approx 0.000308.$$

Inserting this into the equation (2.3) above yields together with $z_{dec} = 1089$ for the time of the decoupling:

$$t_{dec} \approx 379000 \text{ years},$$

in agreement with the result mentioned in the first year WMAP paper.

As a side note, using the dependence of the densities on the scale parameter, we have for the general ratio of the energy densities:

$$\eta = \frac{\eta_0}{a}$$

For $\eta = 1$, we hence have the general result

$$a_{eq} = \eta_0,$$

where a_{eq} was the scale parameter at the time when the energy densities of matter and radiation are equal. Hence we obtain for the redshift at which the shift from radiation- to matter-domination happened:

$$z_{eq} = \frac{1}{a_{eq}} - 1 = \frac{1}{\eta_0} - 1 \approx 24052 \cdot h_0^2 \Omega_{m,0} \approx 3250.$$

This agrees with the formula (15.13) in

http://pdg.lbl.gov/1998/bigbangrpp.pdf.