Proof of formula (A4)

We claimed that if $\tilde{E}_k \neq \epsilon_p$ for all k, then for $N, K \geq 1$

$$\frac{1}{(\omega_f + \epsilon_p + i0^+)^N} \prod_{k=1}^{K} \frac{1}{\omega_f + \tilde{E}_k + i0^+} = \frac{(-1)^{N+1}}{\omega_f + \epsilon_p + i0^+} \sum_{p \in P(N,K)} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} + \text{other terms} \tag{1}$$

holds, where "other terms" refers to summands in which

$$\frac{1}{(\omega_f + \epsilon_p + i0^+)^j}$$

with $j \neq 1$ appears, but no other ω_f-denominators.

Here the set of all ordered partitions with K elements of the integer $N+K-1$ will be denoted by $P(N,K)$, and for $p \in P(N,K)$, p_j means the element number j of the K-tuple p. An ordered partition with K elements of N is a K-tuple for which the sum of the elements gives N.

This formula will be proven here by induction. For $N = K = 1$, we have simply:

$$\begin{aligned}
\frac{1}{\omega_f + \epsilon_p + i0^+} \frac{1}{\omega_f + \tilde{E}_1 + i0^+} &= \frac{1}{\tilde{E}_1 - \epsilon_p} \left(\frac{1}{\omega_f + \epsilon_p + i0^+} - \frac{1}{\omega_f + \tilde{E}_1 + i0^+} \right) \\
&= \frac{1}{\omega_f + \epsilon_p + i0^+} \frac{1}{\tilde{E}_1 - \epsilon_p} + \text{ other terms} \\
&= \frac{(-1)^{1+1}}{\omega_f + \epsilon_p + i0^+} \sum_{p \in P(1,1)} \prod_{k=1}^{1} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&\quad + \text{other terms},
\end{aligned}$$

where a partial fraction decomposition was done and $P(1,1) = (1)$. Hence the formula (1) is right in the case $N = K = 1$.

Consider then first the induction for K, i. e., assume that the formula is right for a given N and all $1, \ldots, K$, and look at the case with $K+1$:

$$\frac{1}{(\omega_f + \epsilon_p + i0^+)^N} \prod_{k=1}^{K+1} \frac{1}{\omega + \tilde{E}_k + i0^+}.$$

Extracting one term from both products and doing a partial fraction decomposition for these terms then gives

$$= \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-1}}\frac{1}{\omega_f+\epsilon_p+i0^+}\frac{1}{\omega_f+\tilde{E}_{K+1}+i0^+}\prod_{k=1}^{K}\frac{1}{\omega_f+\tilde{E}_k+i0^+}$$

$$= \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-1}}\frac{1}{\tilde{E}_{K+1}-\epsilon_p}\left(\frac{1}{\omega_f+\epsilon_p+i0^+}-\frac{1}{\omega_f+\tilde{E}_{K+1}+i0^+}\right)\prod_{k=1}^{K}\frac{1}{\omega_f+\tilde{E}_k+i0^+}$$

This can also be written as

$$\begin{aligned} & \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N}}\frac{1}{\tilde{E}_{K+1}-\epsilon_p}\prod_{k=1}^{K}\frac{1}{\omega_f+\tilde{E}_k+i0^+} \\ - & \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-1}}\frac{1}{\tilde{E}_{K+1}-\epsilon_p}\prod_{k=1}^{K+1}\frac{1}{\omega_f+\tilde{E}_k+i0^+} \end{aligned}$$

The first term above can be rewritten using the induction assumption. In the second, another partial fraction decomposition is done, yielding

$$\begin{aligned} & \frac{(-1)^{N+1}}{\omega_f+\epsilon_p+i0^+}\frac{1}{\tilde{E}_{K+1}-\epsilon_p}\sum_{p\in P(N,K)}\prod_{k=1}^{K}\frac{1}{(\tilde{E}_k-\epsilon_p)^{p_k}} \\ + & \quad \text{other terms} \\ - & \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-1}}\frac{1}{(\tilde{E}_{K+1}-\epsilon_p)^2}\prod_{k=1}^{K}\frac{1}{\omega_f+\tilde{E}_k+i0^+} \\ + & \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-2}}\frac{1}{(\tilde{E}_{K+1}-\epsilon_p)^2}\prod_{k=1}^{K+1}\frac{1}{\omega_f+\tilde{E}_k+i0^+} \end{aligned}$$

The second-to-last term can again be rewritten using the induction assumption, and for the last term, yet another partial fraction decomposition can be done:

$$\begin{aligned} & \frac{(-1)^{N+1}}{\omega_f+\epsilon_p+i0^+}\frac{1}{\tilde{E}_{K+1}-\epsilon_p}\sum_{p\in P(N,K)}\prod_{k=1}^{K}\frac{1}{(\tilde{E}_k-\epsilon_p)^{p_k}} \\ + & \frac{(-1)^{N+1}}{\omega_f+\epsilon_p+i0^+}\frac{1}{(\tilde{E}_{K+1}-\epsilon_p)^2}\sum_{p\in P(N-1,K)}\prod_{k=1}^{K}\frac{1}{(\tilde{E}_k-\epsilon_p)^{p_k}} \\ + & \text{ other terms} \\ + & \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-2}}\frac{1}{(\tilde{E}_{K+1}-\epsilon_p)^2}\prod_{k=1}^{K}\frac{1}{\omega_f+\tilde{E}_k+i0^+} \\ - & \frac{1}{(\omega_f+\epsilon_p+i0^+)^{N-3}}\frac{1}{(\tilde{E}_{K+1}-\epsilon_p)^2}\prod_{k=1}^{K+1}\frac{1}{\omega_f+\tilde{E}_k+i0^+} \end{aligned}$$

Proceeding in the same way, we obtain

$$\frac{1}{(\omega_f+\epsilon_p+i0^+)^{N}}\prod_{k=1}^{K+1}\frac{1}{\omega+\tilde{E}_k+i0^+}$$

$$
\begin{aligned}
&= \frac{(-1)^{N+1}}{\omega + \epsilon_p + i0^+} \sum_{p\in P(N,K)} \frac{1}{\tilde{E}_{K+1} - \epsilon_p} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \frac{(-1)^{N+1}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in P(N-1,K)} \frac{1}{(\tilde{E}_{K+1} - \epsilon_p)^2} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \dots \\
&+ \frac{(-1)^{N+1}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in P(1,K)} \frac{1}{(\tilde{E}_{K+1} - \epsilon_p)^N} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \text{other terms}
\end{aligned}
$$

This can also be written as

$$
\frac{(-1)^{N+1}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in \tilde{P}} \prod_{k=1}^{K+1} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} + \text{other terms},
$$

where $\tilde{P}$ is the following set of ordered partitions of $N + K$ with $K + 1$ elements:

$$
\begin{aligned}
\tilde{P} &= \{(p_1, p_2, \dots, p_K, 1) | p \in P(N, K)\} \\
&\cup \{(p_1, p_2, \dots, p_K, 2) | p \in P(N-1, K)\} \\
&\cup \dots \\
&\cup \{(p_1, p_2, \dots, p_K, N) | p \in P(1, K)\}.
\end{aligned}
$$

But this means that $\tilde{P} = P(N, K+1)$, and thus

$$
\begin{aligned}
\frac{1}{(\omega_f + \epsilon_p + i0^+)^N} \prod_{k=1}^{K+1} \frac{1}{\omega + \tilde{E}_k + i0^+} &= \frac{(-1)^{N+1}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in P(N,K+1)} \prod_{k=1}^{K+1} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \text{other terms},
\end{aligned}
$$

which completes the induction proof for K.

In an analogous way, if we assume that the formula is right for for all $1, \dots, K$ and a given N, it can be shown that for $N + 1$, we have

$$
\begin{aligned}
&\frac{1}{(\omega_f + \epsilon_p + i0^+)^{N+1}} \prod_{k=1}^{K} \frac{1}{\omega + \tilde{E}_k + i0^+} \\
&= \frac{(-1)^{N+2}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in P(N,K)} \frac{1}{\tilde{E}_K - \epsilon_p} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \frac{(-1)^{N+2}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in P(N,K-1)} \frac{1}{(\tilde{E}_K - \epsilon_p)(\tilde{E}_{K-1} - \epsilon_p)} \prod_{k=1}^{K-1} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \dots \\
&+ \frac{(-1)^{N+2}}{\omega_f + \epsilon_p + i0^+} \sum_{p\in P(N,1)} \frac{1}{(\tilde{E}_K - \epsilon_p) \dots (\tilde{E}_1 - \epsilon_p)} \prod_{k=1}^{1} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\
&+ \text{other terms},
\end{aligned}
$$

which also can be rewritten as

$$\frac{(-1)^{N+2}}{\omega_f + \epsilon_p + i0^+} \sum_{p \in \tilde{P}} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} + \text{other terms.}$$

Here, $\tilde{P}$ is the following set of ordered partitions of $N+K$ with K elements:

$$\begin{aligned} \tilde{P} &= \{(p_1, p_2, \ldots, p_K + 1) | p \in P(N, K)\} \\ &\cup \{(p_1, p_2, \ldots, p_{K-1} + 1, 1) | p \in P(N, K-1)\} \\ &\cup \ldots \\ &\cup \{(p_1 + 1, 1, \ldots, 1) | p \in P(N, 1)\}. \end{aligned}$$

It can be seen that $\tilde{P} = P(N+1, K)$, and thus

$$\begin{aligned} \frac{1}{(\omega_f + \epsilon_p + i0^+)^{N+1}} \prod_{k=1}^{K} \frac{1}{\omega + \tilde{E}_k + i0^+} &= \frac{(-1)^{N+2}}{\omega_f + \epsilon_p + i0^+} \sum_{p \in P(N+1,K)} \prod_{k=1}^{K} \frac{1}{(\tilde{E}_k - \epsilon_p)^{p_k}} \\ &+ \text{other terms,} \end{aligned}$$

completing the induction proof for N and thus the proof of (1).